MATH 232: Final Exam 2021 (brief solution)

Exercise 1. There are 10 blue cards numbered from 1 to 10 and 5 green cards numbered from 1 to 5. How many different permutations of the cards can you obtain,

(a) if your eye can distinguish the colors and can see the numbers?

(b) if your eye can distinguish the colors but cannot see the numbers?

(c) if your eye cannot distinguish the colors but can see the numbers?

Sol.: *a)* $15!$ *b)* $\frac{15!}{10!5!}$ *c)* $\frac{15!}{2^5}$

Exercise 2. Assume that a person gets the C-virus when exposed to it with probability
(i) 0.01 if the person is vaccinated and wearing a mask;
(ii) 0.05 if the person is vaccinated and not wearing a mask;
(iii) 0.1 if the person is not vaccinated and wearing a mask;
(iv) 0.5 if the person is not vaccinated and not wearing a mask.
Assume that a person wears a mask with probability 1/2 and that a person is vaccinated with probability 1/3. Furthermore, assume that people wear masks independently of being vaccinated. What is the probability that a person was vaccinated, given that he/she wore a mask and got the C-virus when exposed to it?

Sol.: *We denote:*

$$
\begin{aligned}
&P(I|V, M) = 0.01\\
&P(I|V, M^C) = 0.05\\
&P(I|V^C, M) = 0.1\\
&P(I|V^C, M^C) = 0.5\\
&P(M) = 1/2\\
&P(V) = 1/3.
\end{aligned}
$$

Using Bayes rule and the independence between M and V:

$$P(V|I, M) = \frac{P(I|M, V)P(V|M)}{P(I|M)} = \frac{P(I|M, V)P(V)}{P(I|M)}.$$

Using the law of total probability and plugging in the values we get:

$$P(V|I, M) = \frac{P(I|M, V)P(V)}{P(I|M, V)P(V) + P(I|M, V^C)P(V^C)} = \frac{\frac{1}{100}\cdot\frac{1}{3}}{\frac{1}{100}\cdot\frac{1}{3} + \frac{1}{10}\cdot\frac{2}{3}} = \frac{1}{21}.$$

Exercise 3. Let θ be a random variable that is uniformly distributed in the interval $[-\pi, \pi]$ (this means that the density function of θ equals $\frac{1}{2\pi}$ on the interval $[-\pi, \pi]$ and 0 otherwise). Let $X = \cos(\theta)$ and $Y = \sin(\theta)$.

(a) Are X and Y uncorrelated?

(b) Are X and Y independent?

Sol.:

a) Yes, E[XY] = 0, because $\sin(\theta)\cos(\theta)$ *is an odd function.*
b) No, X can be expressed as a function of Y (or viceversa).

Exercise 4. Assume that in a class of n students, any combination of 3 students forms a friend-triplet (i.e., 3 students who are all friends) with probability e^{-n}. Show that the number of friend-triplets in a class of n students is greater or equal to 1 with vanishing probability as n tends to infinity.

Hint: you can use Markov's inequality to approach this problem.

Sol.: *Let Y_n be the number of friend-triplets in a class of n students. Using Markov's inequality:*

$$P(Y_n > 1) \leq E[Y_n] = \binom{n}{3} e^{-n} \to 0.$$

Exercise 5. Let X_1, X_2 be i.i.d. $\mathcal{N}(0,1)$ and let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

(a) What is the joint distribution of (Y_1, Y_2)?

(b) Are Y_1 and Y_2 independent?

Sol.:

a) $\mathcal{N}_2(\vec{0}, \Sigma)$, *where* $\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
b) Yes. They are jointly Gaussian and uncorrelated (since $Cov(Y_1, Y_2) = 0$).

Exercise 6. Let $Y_1, \ldots, Y_n$ be i.i.d. such that Y_i is uniformly distributed on $[a, b]$, where $a < b$, (this means that the density function of Y_i equals $\frac{1}{b-a}$ on the interval $[a, b]$ and 0 otherwise). Find estimators for a and b using the method of moments.

Hint: you may use the fact that $(x^3 - y^3) = (x - y)(x^2 + xy + y^2)$.

Sol.: *Let $m_1 = \frac{1}{n}\sum_{i=1}^n Y_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^n Y_i^2$. Applying the method of moments, we get*

$$\begin{aligned} m_1 &= \frac{a+b}{2}, \\ m_2 &= \frac{1}{3}(b^2 + ab + a^2). \end{aligned}$$

(The second equation can be analogously written as $\frac{1}{12}(b-a)^2 = s^2 := m_2 - m_1^2$. Computations are even simpler in such case).
From the first equation we get $a = 2m_1 - b$, which we can substitute in the second equation to get

$$\begin{aligned} m_2 &= \frac{1}{3}(b^2 + b(2m_1 - b) + (2m_1 - b)^2) \\ &= \frac{1}{3}(b^2 - 2m_1 b + 4m_1^2). \end{aligned}$$

We can rewrite the last equation as $b^2 - 2m_1b + 4m_1^2 - 3m_2 = 0$, *that has solutions*

$$b_{1,2} = m_1 \pm \sqrt{3(m_2 - m_1^2)}.$$

Replacing this in a, *we get* $a = m_1 \mp \sqrt{3(m_2 - m_1^2)}$. *Since* $a < b$, *we finally get*

$$\begin{aligned} a &= m_1 - \sqrt{3(m_2 - m_1^2)} \\ b &= m_1 + \sqrt{3(m_2 - m_1^2)}. \end{aligned}$$

Exercise 7. Let $Y_1, \ldots, Y_n$ be i.i.d. such that Y_i is uniformly distributed on $[0, \theta]$.

(a) Show that $\max(Y_1, \ldots, Y_n)/\theta$ is a pivot.

(b) Using this pivot, give an equi-tailed two-sided confidence interval for θ with confidence level $1 - \alpha = 7/8$ (i.e. $\alpha = 1/8$) in the case where $n = 2$ and $\max(Y_1, Y_2) = 1$.

Sol.:

a) See Example 267 in slide 339.
Let $M = \max(Y_1, \ldots, Y_n)$,

$$\mathbb{P}(M \leq x\theta) = \mathbb{P}(Y_1 \leq x\theta)^n = \left(\frac{x\theta}{\theta}\right)^n = x^n,$$

that does not depend on θ. *Hence,* M *is a pivot.*

b) See Example 269 in slide 341.
We have $\alpha_L = \alpha_U = 1/16$.

$$\mathbb{P}(\sqrt{\alpha_U} \leq 1/\theta \leq \sqrt{1 - \alpha_L}) = 1 - \alpha_L - \alpha_U = 7/8.$$

So

$$\begin{aligned} L &= 1/\sqrt{1 - \alpha_L} = \frac{4}{\sqrt{15}}, \\ U &= 1/\sqrt{\alpha_U} = 4. \end{aligned}$$

Hence, we obtain the confidence interval $[4/\sqrt{15}, 4]$.

Exercise 8. Consider the binary hypothesis test where a single random variable Y is observed. Under H_0, the random variable Y is uniformly distributed on $[0, 3]$, and under H_1, the random variable Y is uniformly distributed on $[1, 4]$.

(a) Give a test (i.e., define the set of values of Y for which you declare H_0 or H_1) such that the sum of the false negative and false positive probabilities is minimized.

(b) Give a test (i.e., define the set of values of Y for which you declare H_0 or H_1) such that either the false negative or the false positive probability is zero and the other of the two probabilities is minimized.

Sol.: *Y is more likely to take larger values under H_1 than under H_0. A natural test is accept H_0 if $Y \leq t$ and reject H_0 otherwise, for a chosen threshold t.*

a) Choose any $t \in [1, 3]$. One can show that for any $t \notin [1, 3]$, $FP + FN > 2/3$, while $FP + FN = 2/3$ if $t \in [1, 3]$.

b) Either t= 1 or t=3 (depending on whether you impose FP =0 or FN =0).